

Gravitational Collapse of a Massless Scalar Field and a Perfect Fluid with Self-Similarity of the First Kind in (2+1) Dimensions

F. I. M. Pereira¹ *and
R. Chan¹ †

¹ Coordenação de Astronomia e Astrofísica, Observatório Nacional,
Rua General José Cristino 77, São Cristóvão, CEP 20921-400,
Rio de Janeiro, RJ, Brazil

March 19, 2008

Abstract

Self-similar solutions of a collapsing perfect fluid and a massless scalar field with kinematic self-similarity of the first kind in 2+1 dimensions are obtained. Their local and global properties of the solutions are studied. It is found that some of them represent gravitational collapse, in which black holes are always formed, and some may be interpreted as representing cosmological models.

1 Introduction

One of the most interesting problems in gravitation theory is the study of the relation that exists between the critical phenomena and the process of black hole formation. The studies of non-linearity of the Einstein field equations near the threshold of black hole formation reveal very rich phenomena [1]-[3], which are quite similar to critical phenomena in Statistical Mechanics and Quantum Field Theory [4]-[5]. In particular, by numerically studying the gravitational collapse of a massless scalar field in 3 + 1-dimensional spherically symmetric spacetimes, Choptuik found that the mass of such formed black holes takes a scaling form, $M_{BH} = C(p) (p - p^*)^\gamma$, where $C(p)$ is a constant and depends on the initial data, and p parameterizes a family of initial data in such a way that when $p > p^*$ black holes are formed, and when $p < p^*$ no black holes are formed. It was shown that, in contrast to $C(p)$, the exponent γ is universal to all the families of initial data studied. Numerically it was determined as $\gamma \sim 0.37$. The solution with $p = p^*$, usually called the critical solution, is found also universal. Moreover, for the massless scalar field it is periodic, too. Universality of the critical solution and exponent, as well as the power-law scaling of the black hole mass all have given rise to the name *Critical Phenomena in Gravitational Collapse*. Choptuik's studies were soon generalized to other matter fields [6, 7], and now the following seems clear: (a) There are two types of critical collapse, depending on whether the black hole mass takes the scaling form (M_{BH}) or not. When it takes the scaling form, the corresponding collapse is called Type *II* collapse, and when it does not it is called Type *I* collapse. In the type *II* collapse, all the critical solutions found so far have either discrete self-similarity (DSS) or homothetic self-similarity (HSS), depending on the matter fields. In the type *I* collapse, the critical solutions have neither DSS nor HSS. For certain matter fields, these two types of collapse can co-exist. (b) For Type *II* collapse, the corresponding exponent is universal only with respect to certain matter fields. Usually, different matter fields have different critical solutions and, in the sequel, different exponents. But for a given matter field the critical solution and the exponent are universal. So far, the studies have been mainly restricted to spherically symmetric case and it is not clear whether or not the critical solution and exponent are universal with respect to different symmetries of the spacetimes [8, 9, 10]. (c) A critical solution for both of the two types has one and only one unstable mode. This now is considered as one of the main criteria for a solution to be critical. (d) The universality of the exponent is closely related to the last property. In fact, using dimensional analysis [11]-[14] one can show that $\gamma = \frac{1}{|k|}$, where k denotes the unstable mode.

*E-mail Address: flavio@on.br

†E-mail Address: chan@on.br

From the above, one can see that to study (Type *II*) critical collapse, one may first find some particular solutions by imposing certain symmetries, such as, DSS or HSS. Usually this considerably simplifies the problem. For example, in the spherically symmetric case, by imposing HSS symmetry the Einstein field equations can be reduced from PDE's to ODE's. Once the particular solutions are known, one can study their linear perturbations and find out the spectrum of the corresponding eigenmodes. If a solution has one and only one unstable mode, by definition we may consider it as a critical solution (See also the discussions given in [15]). The studies of critical collapse have been mainly numerical so far, and analytical ones are still highly hindered by the complexity of the problem, even after imposing some symmetries.

Lately, Pretorius and Choptuik (PC) [16] studied gravitational collapse of a massless scalar field in an anti-de Sitter background in $2 + 1$ -dimensional spacetimes with circular symmetry, and found that the collapse exhibits critical phenomena and the mass of such formed black holes takes the scaling form (M_{BH}) with $\gamma = 1.2 \pm 0.02$, which is different from that of the corresponding $3 + 1$ -dimensional case. In addition, the critical solution is also different, and, instead of having DSS, now has HSS. The above results were confirmed by independent numerical studies [17]. However, the exponent obtained by Husain and Olivier (HO), $\gamma \sim 0.81$, is quite different from the one obtained by PC. It is not clear whether the difference is due to numerical errors or to some unknown physics.

After the above numerical work, analytical studies of the same problem soon followed up [18, 19, 20, 21, 22, 23]. In particular, Garfinkle found a class, say, $S[n]$, of exact solutions to the Einstein-massless-scalar field equations and showed that in the strong field regime the $n = 4$ solution fits very well with the numerical critical solution found by PC. Lately, Garfinkle and Gundlach (GG) studied their linear perturbations and found that only the solution with $n = 2$ has one unstable mode, while the one with $n = 4$ has three [22]. According to $\gamma = \frac{1}{|k|}$, the corresponding exponent is given by $\gamma = 1/|k| = 4/3$. Independently, Hirschmann, Wang & Wu (HWW) systematically studied the problem, and found that the $n = 4$ solution indeed has only one unstable mode [23]. This difference actually comes from the use of different boundary conditions. As a matter of fact, in addition to the ones imposed by GG [22], HWW further required that no matter field should come out of the already formed black holes. This additional condition seems physically quite reasonable and has been widely used in the studies of black hole perturbations [24]. However, now the corresponding exponent is given by $\gamma = 1/|k| = 4$, which is significantly different from the numerical ones. So far, no explanations about these differences have been worked out, yet.

Self-similarity is usually divided into two classes, one is the discrete self-similarity mentioned above, and the other is the so-called kinematic self-similarity (KSS) [25, 26], and sometimes it is also called continuous self-similarity (CSS). KSS or CSS is further classified into three different kinds, the zeroth, first and second. The kinematic self-similarity of the first kind is also called homothetic self-similarity, first introduced to General Relativity by Cahill and Taub in 1971 [27]. In Statistical Mechanics, critical solutions with KSS of the second kind seem more generic than those of the first kind [4]-[5]. However, critical solutions with KSS of the second kind have not been found so far in gravitational collapse, and it would be very interesting to look for such solutions.

Besides, recently we have studied gravitational collapse of perfect fluid with kinematic self-similarities in four-dimensional spacetimes [28], a subject that has been recently studied intensively (for example, see [29]-[32] and references therein.). In this paper, we shall investigate the same problem but in $2+1$ gravity [33, 34] and including a massless scalar field [35] [36]. The main motivation of such a study comes from recent investigation of critical collapse of a scalar field in 3D gravity [16, 17, 18, 19, 20, 21, 22, 23].

We shall present in this work the self-similar solutions of a collapsing perfect fluid and a massless scalar field with kinematic self-similarity of the first kind in $2+1$ dimensions. Their local and global properties of the solutions are studied. Specifically, in section II we present the self-similar Einstein's field equations, and in section III we show the general solutions obtained. In section IV we present the global properties of the solutions and in Section V we conclude our work.

2 Field Equations

The $2+1$ -dimensional spacetimes with circular symmetry and zero-rotation are described by the metric

$$ds^2 = \gamma_{ab}(t, r)dx^a dx^b + g_{\theta\theta}(t, r)d\theta^2, \quad (1)$$

where $\{x^a\} = \{t, r\}$, ($a, b = 0, 1$), and θ denotes the angular coordinate, with the hypersurfaces $\theta = 0, 2\pi$ being identified. The metric is invariant under the following coordinate transformation,

$$t = t(t', r'), \quad r = r(t', r'). \quad (2)$$

On the other hand, the energy-momentum tensor for a perfect fluid is given by

$$T_{\mu\nu}(\rho, p) = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \quad (3)$$

where u_μ is the velocity of the fluid, ρ and p are its energy density and pressure. The energy-momentum tensor for a massless scalar field is given by

$$T_{\mu\nu}(\phi) = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}, \quad (4)$$

where $(\),_{\alpha} = \partial(\)/\partial_{\alpha}$. We choose the coordinates such that

$$u_\mu \equiv \sqrt{g_{00}} \delta_\mu^0, \quad \text{and} \quad g_{01} = 0, \quad (5)$$

which implies that the coordinates are comoving with the perfect fluid. Then, the metric (1) can be cast in the form

$$ds^2 = l^2 \left\{ e^{2\Phi(t, r)} dt^2 - e^{2\Psi(t, r)} dr^2 - r^2 S^2(t, r) d\theta^2 \right\}, \quad (6)$$

where l is an unit constant with the dimension of length, so that all the coordinates $\{x^a\} = \{t, r, \theta\}$ are dimensionless. We construct a projector operator $h_{\mu\nu}$ by

$$h_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu, \quad (7)$$

from which we find $h^{\alpha\beta}u_\alpha u_\beta = 0$. Once defined the projection operator $h_{\mu\nu}$, following Carter and Henriksen [25, 26], we define the kinematic self-similarity by

$$\mathcal{L}_\xi h_{\mu\nu} = 2h_{\mu\nu}, \quad \mathcal{L}_\xi u^\mu = -\alpha u^\mu, \quad (8)$$

where \mathcal{L}_ξ denotes the Lie differentiation along the vector ξ^μ , α is a *dimensionless* constant. When $\alpha = 0$, the self-similarity is said to be of the *zeroth kind*, when $\alpha = 1$ it is of the *first kind (or homothetic)*, and otherwise it is of the *second kind*. In this work we shall consider only solutions with self-similarity of the first kind ($\alpha = 1$). Applying the above definition to the metric (6), we find that

$$\Phi(t, r) = \Phi(x), \quad \Psi(t, r) = \Psi(x), \quad S(t, r) = S(x), \quad (9)$$

where the self-similar variable x and ξ^μ are given by

$$x = \ln\left(\frac{r}{-t}\right), \quad (10)$$

and

$$\xi^\mu \frac{\partial}{\partial x^\mu} = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}. \quad (11)$$

Before looking for solutions of the Einstein field equations, we must take into account that for the metric to represent circular symmetry some physical and geometrical conditions must be imposed [37]-[46]. We impose the following regularity conditions for the gravitational collapse:

(i) There must exist a symmetry axis, which can be expressed as

$$\mathcal{R} \equiv |\xi_{(\theta)}^\mu \xi_{(\theta)}^\nu g_{\mu\nu}|^{1/2} \rightarrow 0, \quad (12)$$

as $r \rightarrow 0$, where we have chosen the radial coordinate such that the axis is located at $r = 0$, and $\xi_{(\theta)}^\mu$ is the Killing vector with closed orbit given by $\xi_{(\theta)}^\alpha \partial_\alpha = \partial_\theta$.

(ii) The spacetime near the symmetry axis is locally flat, which can be written as [47]

$$\mathcal{R}_{,\alpha} \mathcal{R}_{,\beta} g^{\alpha\beta} \rightarrow -1, \quad (13)$$

as $r \rightarrow 0$. Note that solutions failing to satisfy this condition sometimes are also acceptable. For example, when the right-hand side of the above equation approaches a finite constant, the singularity at $r = 0$ may be related to a point-like particle [48]. However, since here we are mainly interested in gravitational collapse, in this paper we shall assume that this condition holds strictly at the beginning of the collapse, so that we can be sure that the singularity to be formed later on the axis is due to the collapse.

(iii) No closed timelike curves. In spacetimes with circular symmetry, closed timelike curves can be easily introduced. To ensure their absence, we assume that the condition

$$\xi_{(\theta)}^\mu \xi_{(\theta)}^\nu g_{\mu\nu} < 0 , \quad (14)$$

holds in the whole spacetime.

In addition to these conditions, it is usually also required that the spacetime be asymptotically flat in the radial direction. However, since we consider solutions with self-similarity, this condition cannot be satisfied, unless we restrict their validity only up to a maximal radius, say, $r = r_0(t)$, and then join them with others in the region $r > r_0(t)$, which are asymptotically flat as $r \rightarrow \infty$. In this paper, we shall not consider such a possibility, and simply assume that the self-similar solutions are valid in the whole spacetime.

3 Self-Similar Solutions of the First Kind

In this section, we study solutions with self-similarity of the first kind. To obtain the desired equations we substitute equations (3), (4) and (5) into the Einstein field equations

$$G_{\mu\nu} = \kappa [T_{\mu\nu}(\rho, p) + T_{\mu\nu}(\phi)] , \quad (15)$$

and also using the Klein-Gordon equation

$$\square\phi = 0 , \quad (16)$$

where κ is the Einstein coupling constant, $\square = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ with ∇_α being the covariant derivative. When $\alpha = 1$, according to equation (10) terms with powers of r can be substituted by the same powers of $(-t)$ since $r = e^x(-t)$. Then it can be shown that the Einstein field equations in this case become (see Appendix A)

$$e^{2(x-\Phi)} [y\Psi_{,x} - \frac{\kappa}{2}(\phi_{,x} + \phi_{,\tau})^2] - e^{-2\Psi} [y_{,x} + (1+y)(y - \Psi_{,x}) + \frac{\kappa}{2}(\phi_{,x})^2] = \frac{\kappa}{2} \rho_0 , \quad (17)$$

$$-e^{2(x-\Phi)} [y_{,x} + y(y - \Phi_{,x} + 1) + \frac{\kappa}{2}(\phi_{,x} + \phi_{,\tau})^2] + e^{-2\Psi} [(1+y)\Phi_{,x} - \frac{\kappa}{2}(\phi_{,x})^2] = \frac{\kappa}{2} p_0 , \quad (18)$$

$$\begin{aligned} -e^{2(x-\Phi)} [\Psi_{,xx} + \Psi_{,x}(\Psi_{,x} - \Phi_{,x} + 1) + \frac{\kappa}{2}(\phi_{,x} + \phi_{,\tau})^2] &+ \\ e^{-2\Psi} [\Phi_{,xx} + \Phi_{,x}(\Phi_{,x} - \Psi_{,x} - 1) + \frac{\kappa}{2}(\phi_{,x})^2] &= \frac{\kappa}{2} p_0 , \end{aligned} \quad (19)$$

$$y_{,x} + (1+y)(y - \Psi_{,x}) - y\Phi_{,x} + \kappa \phi_{,x}(\phi_{,x} + \phi_{,\tau}) = 0 , \quad (20)$$

$$\begin{aligned} e^{2(x-\Phi)} [(\phi_{,x} + \phi_{,\tau})_{,x} - (\Phi_{,x} - \Psi_{,x} - y - 1)(\phi_{,x} + \phi_{,\tau})] &- \\ e^{-2\Psi} [\phi_{,xx} + (\Phi_{,x} - \Psi_{,x} + y)\phi_{,x}] &= 0 . \end{aligned} \quad (21)$$

In order to determinate the metric coefficients and the x dependence of the scalar field completely, and to make possible the calculations, we assume the equation of state for the perfect fluid given by

$$p = \omega\rho , \quad (22)$$

for any ω .

We use for the massless scalar field the general form $\phi(\tau, x) = a\tau + F(x)$, where $a = \text{constant}$.

To solve the above equations in the general case is not an easy task. In the following, we shall consider several particular cases. We first show some particular solutions for a dust fluid and later for the general case with $\omega \neq 0$. Solutions corresponding to $p = \rho = 0$, that have already been considered in reference [23], are also shown in section 3.3.

3.1 Dust Fluid ($p = 0$)

From geodesic equations we find that $\Phi(x) = \Phi_0$, constant. Then, the Einstein equations are given by

$$e^{2(x-\Phi_0)} \left[y\Psi_{,x} - \frac{\kappa}{2}(\phi_{,x} + \phi_{,\tau})^2 \right] - e^{-2\Psi} \left[y_{,x} + (1+y)(y - \Psi_{,x}) + \frac{\kappa}{2}(\phi_{,x})^2 \right] = \frac{\kappa}{2} \rho_0, \quad (23)$$

$$-e^{2(x-\Phi_0)} \left[y_{,x} + y(y+1) + \frac{\kappa}{2}(\phi_{,x} + \phi_{,\tau})^2 \right] - e^{-2\Psi} \frac{\kappa}{2}(\phi_{,x})^2 = 0, \quad (24)$$

$$-e^{2(x-\Phi_0)} \left[\Psi_{,xx} + \Psi_{,x}(\Psi_{,x} + 1) + \frac{\kappa}{2}(\phi_{,x} + \phi_{,\tau})^2 \right] + e^{-2\Psi} \frac{\kappa}{2}(\phi_{,x})^2 = 0, \quad (25)$$

$$y_{,x} + (1+y)(y - \Psi_{,x}) + \kappa \phi_{,x}(\phi_{,x} + \phi_{,\tau}) = 0, \quad (26)$$

$$e^{2(x-\Phi_0)} \left[(\phi_{,x} + \phi_{,\tau})_{,x} + (\Psi_{,x} + y+1)(\phi_{,x} + \phi_{,\tau}) \right] - e^{-2\Psi} \left[\phi_{,xx} - (\Psi_{,x} - y)\phi_{,x} \right] = 0. \quad (27)$$

In the following, we consider several particular cases.

3.1.1 Solution 1

$$\begin{aligned} \Phi(x) &= \Phi_0 \\ \Psi(x) &= -x + \Psi_0 \\ y(x) &= -1 \\ S(x) &= S_0 e^{-x} \\ \phi(x, \tau) &= \phi_0 \\ \rho &= \frac{e^{-2\Phi_0}}{\kappa l^2(-t)^2} \\ p &= 0 \end{aligned} \quad (28)$$

where Φ_0 , Ψ_0 , S_0 and ϕ_0 are arbitrary integration constants.

3.1.2 Solution 2

$$\begin{aligned} \Phi(x) &= \Phi_0 \\ \Psi(x) &= \ln \left[\cosh \left(\frac{x - x_0}{2} \right) \right] - \frac{x}{2} + \Psi_0 \\ y(x) &= -1 \\ S(x) &= S_0 e^{-x} \\ \phi(x, \tau) &= \phi_0 \\ \rho &= \frac{e^{-2\Phi_0}}{2 \kappa l^2(-t)^2} \left[1 - \tanh \left(\frac{x - x_0}{2} \right) \right] \\ p &= 0 \end{aligned} \quad (29)$$

3.1.3 Solution 3

$$\begin{aligned} \Phi(x) &= \Phi_0 \\ \Psi(x) &= -a \sqrt{\frac{\kappa}{2}} x + \Psi_0 \\ y(x) &= a \sqrt{\frac{\kappa}{2}} - 1 \\ S(x) &= S_0 e^{y-x} \\ \phi(x, \tau) &= a\tau + \phi_0 \\ \rho &= 0 \\ p &= 0, \\ a &= \frac{1}{\sqrt{2K}} \end{aligned} \quad (30)$$

3.2 $p = \omega\rho$

Now, let us consider particular solutions for $\omega \neq 0$. Thus, we will use the general equations (17)-(21).

3.2.1 Solution 4

$$\begin{aligned}
\Phi(x) &= \Phi_0 \\
\Psi(x) &= \frac{-x}{\omega + 1} + \Psi_0 \\
y(x) &= \frac{-1}{\omega + 1} \\
S(x) &= S_0 e^{yx} \\
\phi(x, \tau) &= \phi_0 \\
\rho &= \frac{e^{-2\Phi_0}}{\kappa l^2(\omega + 1)^2(-t)^2} \\
p &= \omega\rho
\end{aligned} \tag{31}$$

3.2.2 Solution 5

$$\begin{aligned}
\Phi(x) &= \Phi_0 \\
\Psi(x) &= -\frac{x}{2} + \Psi_0 \\
y(x) &= -\frac{1}{2} \\
S(x) &= S_0 e^{-x/2} \\
\phi(x, \tau) &= a\tau + \phi_0 \\
\rho &= \frac{e^{-2\Phi_0}}{4\kappa l^2(-t)^2} (1 - 2\kappa a^2) \\
p &= \rho
\end{aligned} \tag{32}$$

3.2.3 Solution 6

$$\begin{aligned}
\Phi(x) &= \pm\Phi_1 \ln|e^{x+2\Psi_0} - 1| + \kappa a^2 x + \Phi_0 \\
\Psi(x) &= \Phi(x) - \frac{x}{2} + \Psi_0 \\
y(x) &= -1/2 \\
S(x) &= S_0 e^{-x/2} \\
\phi(x, \tau) &= a(\tau - x) + \phi_0 \\
\rho &= p = -\frac{e^{-2\Phi_0}(2\kappa a^2 - 1 \pm 2\Phi_1)}{4\kappa l^2(-t)^2(1 - \kappa a^2) r^{2\kappa a^2} |e^{x+2\Psi_0} - 1|^{\pm 2\Phi_1}}
\end{aligned} \tag{33}$$

with + if $x + 2\Psi_0 > 0$ and - if $x + 2\Psi_0 < 0$

3.2.4 Solution 7

$$\begin{aligned}
\Phi(x) &= \Phi_0 \\
\Psi(x) &= -x + \Psi_0, \quad \Psi_0 = \Phi_0 \\
y(x) &= \frac{-1}{1 + \kappa a^2} \\
S(x) &= S_0 e^{yx} \\
\phi(x, \tau) &= a(\tau + yx) + \phi_0 \\
\rho &= \frac{e^{-2\Phi_0}}{\kappa l^2(-t)^2} (1 - y) \frac{1 + 2y}{2y}
\end{aligned} \tag{34}$$

$$\begin{aligned}
p &= \omega\rho \\
\omega &= \frac{1+y}{1-y} \\
a &= \pm\sqrt{-\frac{1+y}{\kappa y}}, \quad -1 \leq y < 0
\end{aligned}$$

3.3 $p = \rho = 0$

3.3.1 Solution 8

$$\begin{aligned}
\Phi(x) &= \Phi_0 \\
\Psi(x) &= \Psi_0 \\
y(x) &= \frac{1}{2}[\tanh(\frac{x-x_0}{2}) - 1] \\
S(x) &= S_0 e^{-x/2} \cosh(\frac{x-x_0}{2}) \\
\phi(x, \tau) &= \phi_0 \\
\rho &= p = 0
\end{aligned} \tag{35}$$

3.3.2 Solution 9

$$\begin{aligned}
\Phi(x) &= \Phi_0 \\
\Psi(x) &= -\frac{x}{2} + \Psi_0 \\
y(x) &= -1/2 \\
S(x) &= S_0 e^{-x/2} \\
\phi(x, \tau) &= a\tau + \phi_0 \\
\rho &= p = 0 \\
a &= \pm\frac{1}{\sqrt{\kappa}}
\end{aligned} \tag{36}$$

3.3.3 Solution 10

$$\begin{aligned}
\Phi(x) &= \Phi_0 \\
\Psi(x) &= -x + \Psi_0, \quad \Psi_0 = \Phi_0 \\
y(x) &= -1/2 \\
S(x) &= S_0 e^{-x/2} \\
\phi(x, \tau) &= a(\tau - x/2) + \phi_0 \\
\rho &= p = 0 \\
a &= \pm\frac{1}{\sqrt{\kappa}}
\end{aligned} \tag{37}$$

3.3.4 Solution 11

$$\begin{aligned}
\Phi(x) &= \frac{1-2\kappa a^2}{2} \ln|e^{x+2\Psi_0} - 1| + \kappa a^2(x + 2\Psi_0) + \Phi_0 \\
\Psi(x) &= \Phi(x) - \frac{x}{2} + \Psi_0 \\
y(x) &= -1/2 \\
S(x) &= S_0 e^{-x/2} \\
\phi(x, \tau) &= a(\tau - x) + \phi_0 \\
\rho &= p = 0
\end{aligned} \tag{38}$$

3.3.5 Solution 12

$$\begin{aligned}
\Phi(x) &= \frac{x}{2} + \Phi_0 \\
\Psi(x) &= -\frac{x}{2} + \Psi_0, \quad \Psi_0 = \Phi_0 \\
y(x) &= -1/2 \\
S(x) &= S_0 e^{-x/2} \\
\phi(x, \tau) &= a(\tau - x/2) + \phi_0 \\
\rho &= p = 0 \\
a &= \pm \frac{1}{\sqrt{\kappa}}
\end{aligned} \tag{39}$$

4 Properties of the Self-Similar Solutions

In this section we shall analyze the global structure for the solutions found in the last section.

4.1 Dust Fluid

Let us first consider the solutions for $p = 0$.

4.1.1 Solution 1

From equation (28), we find that the corresponding metric can be written as

$$ds^2 = l^2 \left\{ e^{2\Phi_0} dt^2 - \frac{(-t)^2}{r^2} \left[e^{2\Psi_0} dr^2 + S_0^2 r^2 d\theta^2 \right] \right\}. \tag{40}$$

The geometrical radius is given by

$$\mathcal{R} = l S_0 r e^{-x} = l S_0 (-t). \tag{41}$$

Without loss of generality, we assume that $S_0 > 0$. Since \mathcal{R} is a function of t only, it is easy to see that $\mathcal{R}_{,t}$ is always positive, since $-t > 0$. The whole spacetime is trapped, as one can see from the outgoing and ingoing null geodesics expansions, which are now given by the equations (see Appendix A)

$$\theta_l = -\frac{e^{-\Phi_0}}{2g(-t)} \quad \text{and} \quad \theta_n = -\frac{e^{-\Phi_0}}{2f(-t)}, \tag{42}$$

from which we have that $\theta_l \theta_n > 0$.

From equation (28) we see that the space-time is always singular when $(-t) \rightarrow 0$. On the other hand, the following expression,

$$R_{,\alpha} R_{,\beta} g^{\alpha\beta} = l^2 S_0^2 e^{-2x} \left[e^{-2\Phi_0} \frac{r^2}{(-t)^2} - 0 \right] = l^2 S_0^2 e^{-2\Phi_0} \tag{43}$$

is always nonzero. In this solution we do not have an apparent horizon but a singularity at $t = 0$. Thus, it may be interpreted as representing a cosmological model.

4.1.2 Solution 2

In the case of equation (29), the metric reads

$$ds^2 = l^2 \left\{ e^{2\Phi_0} dt^2 - \frac{e^{2\Psi_0+x_0}}{4} \left[e^{-x_0} + \frac{(-t)}{r} \right]^2 dr^2 - S_0^2 (-t)^2 d\theta^2 \right\}. \tag{44}$$

The outgoing and ingoing null geodesics expansions

$$\theta_l = -\frac{e^{-\Phi_0}}{2g(-t)} \quad \text{and} \quad \theta_n = -\frac{e^{-\Phi_0}}{2f(-t)} \quad (45)$$

are such that $\theta_l \theta_n > 0$. Thus, there is no apparent horizon.

The geometrical radius is given by

$$R = lS_0 r e^{-x} = lS_0(-t) , \quad (46)$$

where it is assumed that $S_0 > 0$. On the other hand, we see that the right hand side of the expression

$$R_{,\alpha} R_{,\beta} g^{\alpha\beta} = l^2 S_0^2 e^{-2x} \left[e^{-2\Phi_0} \frac{r^2}{(-t)^2} - 0 \right] = l^2 S_0^2 e^{-2\Phi_0} , \quad (47)$$

is always nonzero.

This case is similar to that of Solution 1. Thus, this solution, as in Solution 1, may be interpreted as representing a cosmological model.

4.1.3 Solution 3

We can see that this solution is equal to Solution 9, which is a solution studied by Hirschmann, Wang & Wu [23].

4.2 $p = \omega\rho$

Now, we consider the solutions of equations (31)-(34) for $w \neq 0$.

4.2.1 Solution 4

From equation (31) we see that for $0 \leq \omega \leq 1$ the space-time is the same as that obtained by Miguelote et al. [34], but for a constant scalar field which, without loss of generality, can be $\phi_0 = 0$.

4.2.2 Solution 5

From equation (32), we find that the corresponding metric can be written in the form

$$ds^2 = l^2 \left\{ e^{2\Phi_0} dt^2 - \frac{(-t)}{r} (e^{2\Psi_0} dr^2 + r^2 S_0^2 d\theta^2) \right\} . \quad (48)$$

The geometrical radius is given by

$$\mathcal{R} = lrS_0 e^{-x/2} = lS_0 r^{1/2} (-t)^{1/2} . \quad (49)$$

We assume that $S_0 > 0$, since $\mathcal{R} > 0$.

Let us now investigate the behavior of the ingoing and outgoing expansions of the null geodesics. From equations (32), (70) and (71) we find that

$$\theta_l = \frac{e^{-\Phi_0}}{4g(-t)} \left[\left(\frac{r_{AH}}{r} \right)^{1/2} - 1 \right] \begin{cases} > 0 & \text{if } r < r_{AH} \\ = 0 & \text{if } r = r_{AH} \\ < 0 & \text{if } r > r_{AH} \end{cases} \quad (50)$$

and

$$\theta_n = -\frac{e^{-\Phi_0}}{4f(-t)} \left[\left(\frac{r_{AH}}{r} \right)^{1/2} + 1 \right] < 0 , \quad \text{for any } r \in [0, \infty) . \quad (51)$$

where

$$r_{AH} = e^{2(\Phi_0 - \Psi_0)} (-t) \quad (52)$$

is the apparent horizon. From the above equations we see that

$$\theta_l \theta_n \begin{cases} > 0 & \text{if } r > r_{AH} \text{ -- trapped} \\ = 0 & \text{if } r = r_{AH} \text{ -- marginally trapped} \\ < 0 & \text{if } r < r_{AH} \text{ -- untrapped} \end{cases} \quad (53)$$

In order to satisfy the regularity condition (13) we find that

$$S_0 = 2e^{\Psi_0} . \quad (54)$$

Thus, we conclude that we have a black hole.

4.2.3 Solution 6

Due to the discontinuities in ρ and θ_n at the horizon, this solution does not represent a physical system.

4.2.4 Solution 7

From equation (34) we find that the metric is given by

$$ds^2 = l^2 \left\{ e^{2\Psi_0} dt^2 - \frac{(-t)^2}{r^2} e^{2\Psi_0} dr^2 - S_0^2 \frac{r^{2(1+y)}}{(-t)^{2y}} d\theta^2 \right\} . \quad (55)$$

The geometrical radius is given by

$$\mathcal{R} = lrS_0 e^{yx} = lS_0 \frac{r^{1+y}}{(-t)^y} , \quad (56)$$

where we can see that $\mathcal{R} \rightarrow 0$ when $r \rightarrow 0$, since $-1 < y < 0$ because a is real. We also assume $S_0 > 0$.

The outgoing and ingoing null geodesics expansions are given by

$$\theta_l = \frac{e^{-\Psi(x)}}{2gr} \left(\frac{3\omega - 1}{\omega + 1} \right) \begin{cases} > 0 & \text{if } \omega > 1/3 \ (y > -1/2) \\ = 0 & \text{if } \omega = 1/3 \ (y = -1/2) \\ < 0 & \text{if } \omega < 1/3 \ (y < -1/2) \end{cases} \quad (57)$$

and

$$\theta_n = -\frac{e^{-\Psi(x)}}{2fr} < 0 \text{ for any } r \in [0, \infty) . \quad (58)$$

Thus, this solution does not present an apparent horizon and it represents a cosmological solution.

4.2.5 Solutions 8

As can be seen in equation (35), this solution, corresponds to a flat space-time.

4.2.6 Solutions 9, 10, 11 e 12

Since these solutions have already been studied by Hirschmann, Wang & Wu [23], we do not present any analysis of them.

5 Conclusions

In this paper we have obtained self-similar solutions of the Einstein field equations for a collapsing massless scalar field and perfect fluid with kinematic self-similarity of the first kind in 2+1 dimensions. The local and global properties of the solutions are studied. Since in the general case it is not an easy task to solve the system of field equations, we have considered some particular cases. We have found 12 solutions. One of them does not represent a physical system (Solution 6) and another one of them represents a Minkowski spacetime (Solution 8). Six of them represent solutions already studied by others authors (Solutions 3, 4, 9, 10, 11 and 12). Thus, we have obtained three new cosmological solutions (Solutions 1, 2 and 7) and a new black hole solution (Solution 5).

Acknowledgments

We would like to thank Dr. Anzhong Wang for the useful and helpful discussions about this work. One of the authors (RC) acknowledges the financial support from FAPERJ (no. E-26/171.754/2000 and E-26/171.533/2002) and from Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq - Brazil.

Appendix A - The field equations

The non-zero components of the Einstein tensor are given by

$$G_{tt} = \frac{1}{(-t)^2} y \Psi_{,x} - \frac{e^{2(\Phi-\Psi)}}{r^2} [y_{,x} + (1+y)(y - \Psi_{,x})] \quad (59)$$

$$G_{rr} = \frac{-e^{2(\Psi-\Phi)}}{(-t)^2} [y_{,x} + y(y - \Phi_{,x} + 1)] + \frac{1}{r^2} (1+y) \Phi_{,x} \quad (60)$$

$$\begin{aligned} G_{\theta\theta} &= S^2 r^2 \left\{ -\frac{e^{-2\Phi}}{(-t)^2} [\Psi_{,xx} + \Psi_{,x}(\Psi_{,x} - \Phi_{,x} + 1)] \right. \\ &\quad \left. + \frac{e^{-2\Psi}}{r^2} [\Phi_{,xx} + \Phi_{,x}(\Phi_{,x} - \Psi_{,x} - 1)] \right\} \end{aligned} \quad (61)$$

$$G_{tr} = \frac{-1}{(-t) r} [y_{,x} + (1+y)(y - \Psi_{,x}) - y \Phi_{,x}] , \quad (62)$$

where

$$y \equiv \frac{S_{,x}}{S} = (\ln S)_{,x} , \quad (63)$$

and the Klein-Gordon equation can be written as

$$\begin{aligned} &\frac{e^{-2\Phi}}{(-t)^2} [\phi_{,xx} - (\Phi_{,x} - \Psi_{,x} - y - 1)(\phi_{,x} + \phi_{,\tau})] \\ &- \frac{e^{-2\Psi}}{r^2} [\phi_{,xx} + (\Phi_{,x} - \Psi_{,x} + y)\phi_{,x}] = 0 . \end{aligned} \quad (64)$$

The non-zero components of the energy-momentum tensor for a massless scalar field and a perfect fluid are given by

$$T_{tt} = \frac{1}{2(-t)^2} (\phi_{,x} + \phi_{,\tau})^2 + \frac{e^{2(\Phi-\Psi)}}{2r^2} (\phi_{,x})^2 + \rho e^{2\Phi} \quad (65)$$

$$T_{rr} = \frac{e^{2(\Psi-\Phi)}}{2(-t)^2} (\phi_{,x} + \phi_{,\tau})^2 + \frac{1}{2r^2} (\phi_{,x})^2 + p e^{2\Psi} \quad (66)$$

$$T_{\theta\theta} = r^2 S^2 \left[\frac{e^{-2\Phi}}{2(-t)^2} (\phi_{,x} + \phi_{,\tau})^2 - \frac{e^{-2\Psi}}{2r^2} (\phi_{,x})^2 + p \right] \quad (67)$$

$$T_{tr} = \frac{1}{r(-t)} \phi_{,x} (\phi_{,x} + \phi_{,\tau}) . \quad (68)$$

In order to put the field equations in a more suitable form for calculations we write the energy density and pressure as

$$\rho \equiv \frac{\rho_0(x)}{2r^2} \quad \text{and} \quad p \equiv \frac{p_0(x)}{2r^2} . \quad (69)$$

For the self-similar solutions of the first kind the outgoing and ingoing expansion of the null geodesics and $\mathcal{R}_{,\alpha} \mathcal{R}_{,\beta} g^{\alpha\beta}$ are, respectively, given by [34]

$$\theta_l = \frac{1}{2rg} \left\{ (1+y)e^{-\Psi(x)} + ye^{x-\Phi(x)} \right\}, \quad (70)$$

$$\theta_n = \frac{-1}{2rf} \left\{ (1+y)e^{-\Psi(x)} - ye^{x-\Phi(x)} \right\}, \quad (71)$$

where f and g are assumed to be positive functions, $f > 0$ and $g > 0$ (for a more detailed discussion, see appendix B of reference [34]), and

$$\mathcal{R}_{,\alpha} \mathcal{R}_{,\beta} g^{\alpha\beta} = S^2(x) \left\{ y^2 \frac{r^2 e^{-2\Phi(x)}}{(-t)^2} - (1+y)^2 e^{-2\Psi(x)} \right\}. \quad (72)$$

References

- [1] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
- [2] M. W. Choptuik, “Critical Behavior in Massless Scalar Field Collapse,” in *Approaches to Numerical Relativity, Proceedings of the International Workshop on Numerical Relativity*, Southampton, December, 1991, Edited by Ray d’Inverno.
- [3] M. W. Choptuik, “Critical Behavior in Scalar Field Collapse,” in *Deterministic Chaos in General Relativity*, Edited by D. Hobill et al. (Plenum Press, New York, 1994), p. 155-175.
- [4] G.I. Barenblatt, *Similarity, Self-Similarity, and Intermediate Asymptotics* (Consultants Bureau, New York, 1979).
- [5] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison Wesley Publishing Company, New York, 1992).
- [6] C. Gundlach, “Critical phenomena in gravitational collapse: Living Reviews,” [gr-qc/0001046](https://arxiv.org/abs/gr-qc/0001046) (2000), and references therein.
- [7] A. Wang, “Critical Phenomena in Gravitational Collapse: The Studies So Far,” [gr-qc/0104073](https://arxiv.org/abs/gr-qc/0104073), Braz. J. Phys. **31**, 188 (2001), and references therein.
- [8] M.W. Choptuik, E.W. Hirschmann, S.L. Liebling, and F. Pretorius, Phys. Rev. **D68**, 044007 (2003).
- [9] M.W. Choptuik, E.W. Hirschmann, S.L. Liebling, and F. Pretorius, Phys. Rev. Lett. **93**, 131101 (2004).
- [10] A.Z. Wang, Phys. Rev. **D68**, 064006 (2003).
- [11] C. R. Evans and J. S. Coleman, Phys. Rev. Lett. **72**, 1782 (1994).
- [12] T. Koike, T. Hara, and S. Adachi, Phys. Rev. Lett., **74**, 5170 (1995).
- [13] C. Gundlach, Phys. Rev. Lett. **75**, 3214 (1995).
- [14] E. W. Hirschmann and D. M. Eardley, Phys. Rev. **D52**, 5850 (1995).
- [15] P. R. Brady, M. W. Choptuik, C. Gundlach, and D. W. Neilsen, Class. Quantum Grav. **19**, 6359 (2002).
- [16] F. Pretorius and M. W. Choptuik, Phys. Rev. **D62**, 124012 (2000).
- [17] V. Husain and M. Olivier, Class. Quantum Grav. **18**, L1 (2001).
- [18] D. Garfinkle, Phys. Rev. **D63**, 044007 (2001).
- [19] G. Clément and A. Fabbri, Class. Quantum Grav. **18**, 3665 (2001).
- [20] G. Clément and A. Fabbri, Nucl. Phys. **B630**, 269 (2002).
- [21] M. Cavaglia, G. Clement, and A. Fabbri, Phys. Rev. **D70**, 044010 (2004).

- [22] D. Garfinkle and C Gundlach, Phys. Rev. **D66**, 044015 (2002).
- [23] E.W. Hirschmann, A.Z. Wang, and Y. Wu, Class. Quant. Grav. **21**, 1791 (2004).
- [24] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon Press, Oxford University Press, Oxford, 1983).
- [25] B. Carter and R.N. Henriksen, Ann. Physique Suppl. **14**, 47 (1989).
- [26] A.A. Coley, Class. Quantum Grav. **14**, 87 (1997).
- [27] M.E. Cahill and A.H. Taub, Commun. Math. Phys. **21**, 1 (1971).
- [28] C.F. Brandt, L.-M. Lin, J. F. Villas da Rocha and A.Z. Wang, Inter. J. Mod. Phys. **D11**, 155 (2002).
- [29] B. J. Carr and A. A. Coley, Class. Quantum Grav. **16**, R31 (1999).
- [30] H.-C. Kim, S.-H. Moon, and J. H. Yee, JHEP, **0202**, 046 (2002).
- [31] H. Maeda, T. Harada, H. Iguchi, N. Okuyama, Prog. Theor. Phys. **108**, 819 (2002).
- [32] H. Maeda, T. Harada, H. Iguchi, N. Okuyama, Prog. Theor. Phys. **110**, 25 (2003).
- [33] S. Carlip, *Quantum Gravity in 2 + 1 Dimensions* (Cambridge University Press, Cambridge, 1998).
- [34] A. Y. Miguelote, N. A. Tomimura, and A.Z. Wang, Gen. Rel. Grav. **36**, 1883 (2004).
- [35] F. I. M. Pereira, R. Chan , A. Z. Wang, Int. J. Mod. Phys. D, **15**, 131 (2006).
- [36] R. Chan, M.F.A. da Silva, J.F. Villas da Rocha, A. Wang, Int. J. Mod. Phys. D, **14**, 1049 (2005).
- [37] M. Mars and J.M.M. Senovilla, Class. Quantum Grav. **10**, 1633 (1993).
- [38] P.R.C.T. Pereira, N.O. Santos, and A.Z. Wang, Class. Quantum Grav. **13**, 1641 (1996).
- [39] M.A.H. MacCallum and N.O. Santos, Class. Quantum Grav. **15**, 1627 (1998).
- [40] M.A.H. MacCallum, Gen. Relativ. Grav. **30**, 131 (1998).
- [41] P.R.C.T. Pereira, A.Z. Wang, Gen. Relativ. Grav. **32**, 2189 (2000).
- [42] J. Carot, J.M.M. Senovilla, and R. Vera, Class. Quantum Grav. **16**, 3025 (1999).
- [43] A. Barnes, Class. Quantum Grav. **17**, 2605 (2000).
- [44] L. Herrera, N.O. Santos, A.F.F. Teixeira, and A.Z. Wang, Class. Quantum Grav. **18**, 3847 (2001).
- [45] A.Y. Miguelote, M.F.A. da Silva, A.Z. Wang, and N.O. Santos, Class. Quantum Grav. **18**, 4569 (2001).
- [46] M.F.A. da Silva, L. Herrera, N.O. Santos, and A.Z. Wang, Class. Quantum Grav. **19**, 3809 (2002).
- [47] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).
- [48] A. Vilenkin and E. P. S. Shellard, *Cosmic String and Other Topological Defects* (Cambridge University Press, Cambridge, 1994).